Autonomous Navigation for Flying Robots

Lecture 5.2: Recap on Probability Theory

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Probability Theory

- Random experiment that can produce a number of outcomes, e.g., rolling a dice
- Sample space, e.g., \( \{1, 2, 3, 4, 5, 6\} \)
- Event \( A \) is subset of outcomes, e.g., \( \{2, 4, 6\} \)
- Probability \( P(A) \), e.g., \( P(A) = 0.5 \)
Axioms of Probability Theory

1. \(0 \leq P(A) \leq 1\)

2. \(P(\Omega) = 1\) \(P(\emptyset) = 0\)

3. \(P(A \cup B) = P(A) + P(B) - P(A \cap B)\)
A Closer Look at Axiom 3

\[ P(A \cup B) = P(A) + P(B) - P(A \cap B) \]
Discrete Random Variables

- $X$ denotes a random variable
- $X$ can take on a countable number of values in $\{x_1, x_2, \ldots x_n\}$
- $P(X = x_i)$ is the probability that the random variable $X$ takes on value $x_i$
- $P(\cdot)$ is called the probability mass function

Example: $P(\text{Room}) = \langle 0.7, 0.2, 0.08, 0.02 \rangle$

\[
\text{Room} \in \{\text{office, corridor, lab, kitchen}\}
\]
Continuous Random Variables

- $X$ takes on continuous values
- $p(X = x)$ or $p(x)$ is called the **probability density function (PDF)**

$$P(x \in [a, b]) = \int_a^b p(x) \, dx$$

**Example**

[Diagram showing probability density function and intervals]

http://mitpress.mit.edu/books/probabilistic-robotics
Proper Distributions Sum To One

- Discrete case

\[ \sum_{x} P(x) = 1 \]

- Continuous case

\[ \int p(x) \, dx = 1 \]
Joint and Conditional Probabilities

- $P(X = x \text{ and } Y = y) = P(x, y)$

- If $X$ and $Y$ are independent then
  
  $$P(x, y) = P(x)P(y)$$

- $P(x \mid y)$ is the probability of $x$ given $y$
  
  $$P(x \mid y)P(y) = P(x, y)$$

- If $X$ and $Y$ are independent then
  
  $$P(x \mid y) = P(x)$$
Conditional Independence

- Definition of conditional independence

\[ P(x, y \mid z) = P(x \mid z)P(y \mid z) \]

- Equivalent to

\[ P(x \mid z) = P(x \mid y, z) \]
\[ P(y \mid z) = P(y \mid x, z) \]

- Note: this does not necessarily mean that

\[ P(x, y) = P(x)P(y) \]
Marginalization

- Discrete case
  \[ P(x) = \sum_{y} P(x, y) \]

- Continuous case
  \[ p(x) = \int p(x, y) dy \]
Example: Marginalization

<table>
<thead>
<tr>
<th>( P(X,Y) )</th>
<th>( x_1 )</th>
<th>( x_2 )</th>
<th>( x_3 )</th>
<th>( x_4 )</th>
<th>( P(Y) )</th>
</tr>
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<tbody>
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<td>( y_1 )</td>
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<td>1/16</td>
<td>1/32</td>
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<tr>
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<td>0</td>
<td>0</td>
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<td>1/4</td>
<td>1/8</td>
<td>1/8</td>
<td>1</td>
</tr>
</tbody>
</table>
Expected Value of a Random Variable

- **Discrete case**
  \[ E[X] = \sum_{i} x_i P(x_i) \]

- **Continuous case**
  \[ E[X] = \int x P(X = x) \, dx \]

- The expected value is the weighted average of all values a random variable can take on.

- Expectation is a linear operator
  \[ E[aX + b] = aE[X] + b \]
Covariance of a Random Variable

- Measures the squared expected deviation from the mean

\[ \text{Cov}[X] = E[X - E[X]]^2 = E[X^2] - E[X]^2 \]
Estimation from Data

- Observations \( x_1, x_2, \ldots, x_n \in \mathbb{R}^d \)

- Sample Mean \( \mu = \frac{1}{n} \sum_i x_i \)

- Sample Covariance

\[
\Sigma = \frac{1}{n - 1} \sum_i (x_i - \mu)(x_i - \mu)^\top
\]
Lessons Learned

- Recap on probability theory
- Random variables
- Joint and conditional probabilities
- Marginalization
- Mean and covariance

Next:
Bayes Law and examples