Autonomous Navigation for Flying Robots

Lecture 2.2:
2D Geometry

Jürgen Sturm
Technische Universität München

Geometric Primitives in 2D

- 2D point $$\mathbf{x} = \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2$$
- Augmented vector $$\bar{\mathbf{x}} = \begin{pmatrix} x \\ y \\ 1 \end{pmatrix} \in \mathbb{R}^3$$
- Homogeneous coordinates $$\tilde{\mathbf{x}} = \begin{pmatrix} \tilde{x} \\ \tilde{y} \\ \tilde{w} \end{pmatrix} \in \mathbb{P}^2$$

Homogeneous Vectors

- Homogeneous vectors that differ only by scale represent the same 2D point
- Convert back to inhomogeneous coordinates by dividing through last element

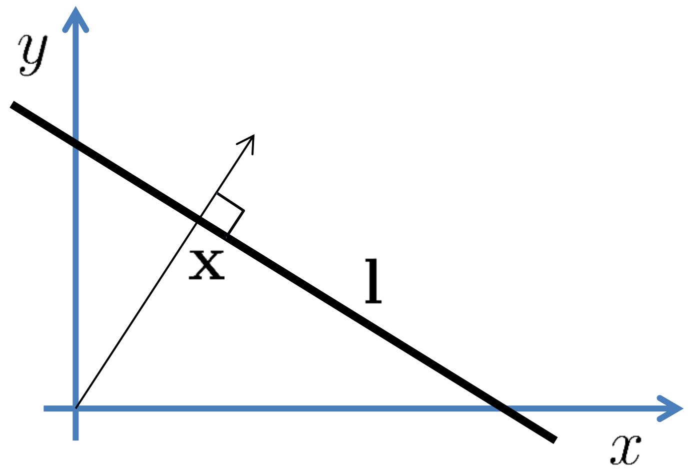
$$\tilde{\mathbf{x}} = \begin{pmatrix} \tilde{x} \\ \tilde{y} \\ \tilde{w} \end{pmatrix} = \tilde{w} \begin{pmatrix} \tilde{x}/\tilde{w} \\ \tilde{y}/\tilde{w} \\ 1 \end{pmatrix} = \tilde{w} \begin{pmatrix} x \\ y \\ 1 \end{pmatrix} = \tilde{w}\bar{\mathbf{x}}$$

- Points with $\tilde{w} = 0$ are called points at infinity or ideal points

Geometric Primitives in 2D

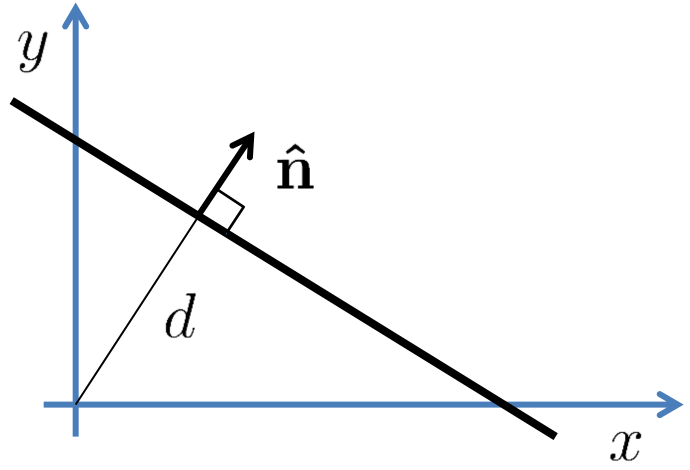
- 2D line $\tilde{\mathbf{l}} = (a, b, c)^\top$

- 2D line equation $\bar{\mathbf{x}} \cdot \tilde{\mathbf{l}} = ax + by + c = 0$



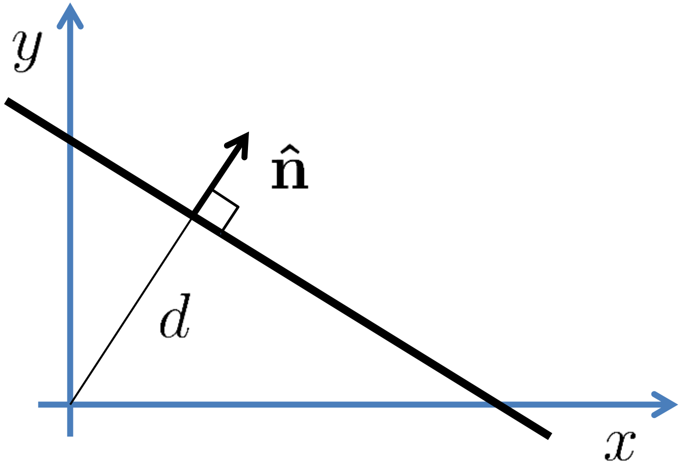
Geometric Primitives in 2D

- Normalized line equation $\tilde{\mathbf{l}} = (\hat{n}_x, \hat{n}_y, d)^\top = (\hat{\mathbf{n}}, d)^\top$
  where $\|\hat{\mathbf{n}}\| = 1$
  and d is the distance of the line to the origin



Geometric Primitives in 2D

- Line joining two points $\tilde{\mathbf{l}} = \tilde{\mathbf{x}}_1 \times \tilde{\mathbf{x}}_2$
- Intersection point of two lines $\tilde{\mathbf{x}} = \tilde{\mathbf{l}}_1 \times \tilde{\mathbf{l}}_2$

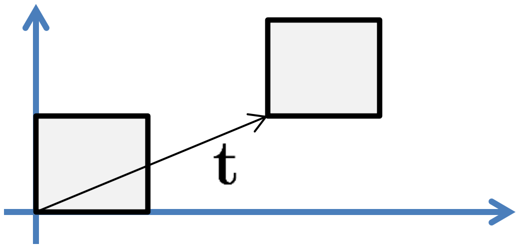


2D Transformations

- Translation $\mathbf{x}' = \mathbf{x} + \mathbf{t}$

$$\mathbf{x}' = \underbrace{\begin{pmatrix} \mathbf{I} & \mathbf{t} \end{pmatrix}}_{2\times 3} \bar{\mathbf{x}}$$

$$\tilde{\mathbf{x}}' = \underbrace{\begin{pmatrix} \mathbf{I} & \mathbf{t} \\ \mathbf{0}^\top & 1 \end{pmatrix}}_{3\times 3} \tilde{\mathbf{x}}$$



where $\mathbf{t} \in \mathbb{R}^2$ is the translation vector, $\mathbf{I}$ is the identity matrix, and $\mathbf{0}$ is the zero vector

2D Transformations

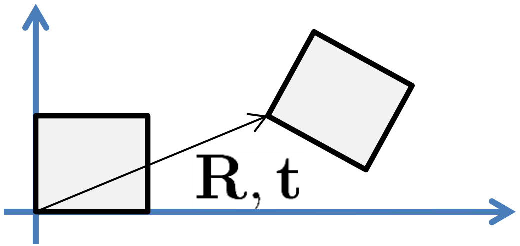
- Rigid body motion or Euclidean transf. (rotation + translation)

$$\mathbf{x}' = \mathbf{R}\mathbf{x} + \mathbf{t} \qquad \text{or} \qquad \tilde{\mathbf{x}}' = \begin{pmatrix} \mathbf{R} & \mathbf{t} \\ \mathbf{0}^\top & 1 \end{pmatrix} \tilde{\mathbf{x}}$$

where $\mathbf{R} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$

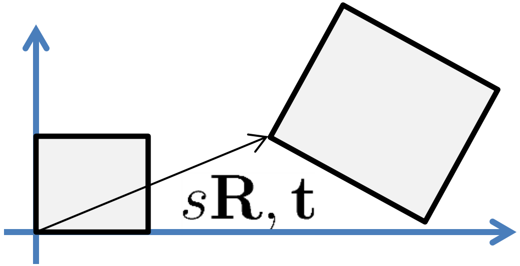
is an orthogonal matrix, i.e., $\mathbf{R}\mathbf{R}^\top = \mathbf{I}$

- Distances (and angles) are preserved



2D Transformations

- Scaled rotation/similarity transform



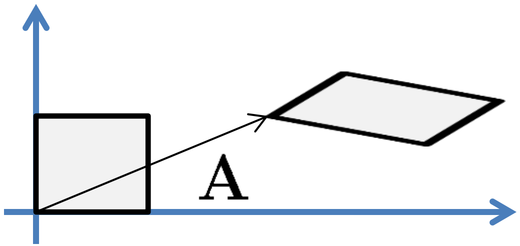
$$\mathbf{x}' = s\mathbf{R}\mathbf{x} + \mathbf{t} \quad \text{or} \quad \tilde{\mathbf{x}}' = \begin{pmatrix} s\mathbf{R} & \mathbf{t} \\ \mathbf{0}^\top & 1 \end{pmatrix} \tilde{\mathbf{x}}$$

- Preserves angles between lines

2D Transformations

- Affine transform

$$\tilde{\mathbf{x}}' = \mathbf{A}\tilde{\mathbf{x}} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ 0 & 0 & 1 \end{pmatrix} \tilde{\mathbf{x}}$$

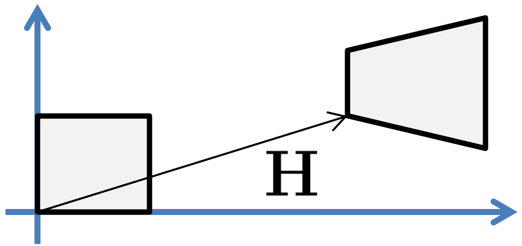


- Parallel lines remain parallel

2D Transformations

- Homography or projective transf.

$$\tilde{\mathbf{x}}' = \tilde{\mathbf{H}}\tilde{\mathbf{x}} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \tilde{\mathbf{x}}$$



- Note that $\tilde{\mathrm{H}}$ is homogeneous (only defined up to scale)
- Resulting coordinates are homogeneous
- Straight lines remain straight

Lessons Learned

- 2D points and 2D lines
- Homogeneous coordinates
- 2D transformations

- Next: Example